



Kyiv School of Economics
founded by EERC and the Victor Pivchuk Foundation



Kyiv School of Economics & Kyiv Economics Institute

DISCUSSION PAPER SERIES

Domain L-Majorization and Equilibrium Existence in Discontinuous Games

Pavlo Prokopovych

Kyiv School of Economics and Kyiv Economics Institute

DP# 31

July 2010

Domain L -Majorization and Equilibrium Existence in Discontinuous Games¹

Pavlo Prokopovych²

Abstract We study the equilibrium existence problem in normal form and qualitative games in which it is possible to associate with each nonequilibrium point an open neighborhood and a collection of deviation strategies such that, at any nonequilibrium point of the neighborhood, a player can increase her payoff by switching to the deviation strategy designated for her. An equilibrium existence theorem for compact, quasiconcave games with two players is established. We propose a new form of the better-reply security condition, called the strong single deviation property, that covers games whose set of Nash equilibria is not necessarily closed.

We introduce domain L -majorized correspondences and use them to study equilibrium existence in qualitative games.

Keywords Better-reply secure game; Discontinuous game; Single deviation property; Majorized correspondence; Qualitative game

JEL classification Numbers C65; C72

Date June 1, 2011

¹I am grateful to Roy Gardner, Yuriy Gorodnichenko, Nicholas Yannelis, and anonymous referees for helpful discussions, comments, and suggestions. Any remaining mistakes are my own.

²Email: prokop@kse.org.ua, pprokopo@gmail.com. Phone: 38-044-492-8012, Fax: 38-044-492-8011, Address: Kyiv School of Economics and Kyiv Economics Institute, 13 Yakira, Office 327, Kyiv 04119, Ukraine

1 Introduction

A number of generalizations and strengthenings of Reny's equilibrium existence theorem for better-reply secure games have been proposed recently. Among them are papers by Barelli and Soza (2009), Carmona (2011), de Castro (2011), McLennan, Monteiro, and Tourky (2009), and Reny (2009).

In this paper, we look at the equilibrium existence problem in normal form and qualitative games through the prism of a property called by Reny (2009) the single deviation property. Both better-reply secure games and diagonally transfer continuous games (Baye, Tian, Zhou, 1993) possess this property. According to it, if a strategy profile is not a Nash equilibrium, then there exist an open neighborhood and a full profile of deviation strategies – one for each player – such that, at any point of the neighborhood, a player can increase her payoff by switching to her deviation strategy. Intuitively, if the single deviation property holds and the game has no Nash equilibrium in pure strategies, we can associate with each strategy profile a neighborhood and a collection of constant valued correspondences defined on the neighborhood. With the help of a partition of unity and an assumption imposed on the convex hulls of deviation strategies, Barelli and Soza (2009, Theorem 2.2) glue the locally defined correspondences together into an upper hemicontinuous correspondence, defined on the Cartesian product of the players' strategy sets, to which Kakutani's fixed point theorem can be applied.

As is shown by a three player example in Reny (2009, Section 3), replacing the better-reply security condition with the single deviation property does not result in a complete set of sufficient conditions for the existence of a pure strategy Nash equilibrium in compact, quasiconcave games. In that example, even though it is possible to find, for every point, a neighborhood and a collection of constant valued correspondences defined on the neighborhood, one cannot glue them together into a well behaved correspondence having the Cartesian product of the players' strategy sets as its domain. Nessah and Tian (2010) show that if, instead of the quasiconcavity

condition, a property related to but stronger than the diagonal transfer quasiconcavity condition (Baye, Tian, and Zhou, 1993) is assumed, then the existence of an equilibrium follows.

The weak single deviation property, introduced in this paper, is weaker than the single deviation property in two respects: (1) deviation strategies need not be defined for all players, and (2) every neighborhood of a nonequilibrium point may contain equilibrium points, as in a second-price sealed-bid auction with two bidders having different valuations.

Intuitively, if a single player can increase her payoff using the same deviation strategy at every point of an open neighborhood of a nonequilibrium point, then having to define deviation strategies for the rest of the players makes the above-mentioned assumption on the convex hulls of deviation strategies less tractable. On the other hand, if deviation strategies are not necessarily defined for all players, then it is impossible to use a partition of unity as the glueing technique. Therefore, there is a need for applying different tools, such as majorized correspondences. A strengthening of Barelli and Soza's equilibrium existence theorem, Theorem 5, and its applications are studied in the last section of the paper, Section 4.

Theorem 3 states that every compact, quasiconcave, two-person game with the weak single deviation property has a pure strategy Nash equilibrium if the players' strategy sets lie on the real line. To show Theorem 3, we proceed by contradiction, assuming that the game has no equilibrium. Then we construct an open cover of the Cartesian product of the players' strategy sets that satisfies the conditions of Theorem 2, a version of Theorem 5 for normal form games.

In Section 3, we introduce a strengthening of the weak single deviation property, the strong single deviation property. This property is not a generalization of the better-reply security condition, but another, slightly improved, form of it. Lemmas 1 and 2 show that the strong single deviation property is equivalent to the better-reply security condition in compact games with no Nash equilibrium in pure strategies. As we demonstrate by example, the strong single deviation property makes it possible to apply Reny's equilibrium existence theorem to games with a noncompact set of pure strategy Nash equilibria. In Remark 1, we describe a possible way of amending

the notion of a better-reply secure game.

Majorized correspondences have served as a powerful tool for analyzing qualitative and generalized games since the groundbreaking works by Borglin and Keiding (1976) and Yannelis and Prabhakar (1983). At the heart of the proof of the above-mentioned Theorem 5 lies the notion of a domain L -majorized correspondence, a generalization of Yannelis and Prabhakar's notion of an L -majorized correspondence. Implicitly, the idea of domain L -majorization has been present in the literature studying majorized correspondences and their applications for quite a while. So, Yuan (1999) introduces into consideration a correspondence whose values majorize the values of the correspondence under study and that has a multivalued selection with open lower sections. This very idea stands behind L_{FC} -majorized correspondences (Ding and Xia 2004). Domain L -majorization, introduced in Section 4, goes a little farther: we do not majorize the values of the correspondence under study, only its domain.

Lemma 5 provides a set of sufficient conditions for a correspondence to be domain L -majorized that are equivalent, in the context of qualitative games, to Barelli and Soza's equilibrium existence conditions (Corollary 3).

Intuitively, the main result of Section 4, Theorem 5, deals with qualitative games having a generalized weak single deviation property. Some of its applications are provided there. Corollary 4 is a version of the Fan-Browder collective fixed point theorem, and Corollary 5 is an equilibrium existence theorem for qualitative games.

2 The Weak Single Deviation Property

Consider a game G between n players where each player i 's pure strategy set X_i is a nonempty, compact subset of a Hausdorff topological vector space, and each payoff function u_i is a bounded function from $X = \prod_{i \in N} X_i$ to \mathbb{R} (X being endowed with the product topology). Under these conditions, $G = (X_i, u_i)_{i \in N}$ is called a compact game. As before, denote the set of players by $N = \{1, \dots, n\}$. A game $G = (X_i, u_i)_{i \in N}$ is quasiconcave if each X_i is convex and $u_i(\cdot, x_{-i}) : X_i \rightarrow \mathbb{R}$ is

quasiconcave for all $i \in N$ and all $x_{-i} \in X_{-i}$, where $X_{-i} = \prod_{k \in N \setminus \{i\}} X_k$. Denote by E_G the set of all pure strategy Nash equilibria of G in X .

Definition 1 *Player i can secure a payoff of $\alpha \in \mathbb{R}$ at $x \in X$ if there exists $d_i \in X_i$ such that $u_i(d_i, x'_{-i}) \geq \alpha$ for all x'_{-i} in some open neighborhood of x_{-i} .*

The graph of G is defined by $\text{Gr}G = \{(x, u) \in X \times \mathbb{R}^n \mid u_i(x) = u_i \text{ for all } i \in N\}$. For a subset B of a topological space X , we denote the interior of B in X by $\text{int}_X B$, the boundary of B by ∂B , and the closure of B by $\text{cl}B$.

Definition 2 *A game $G = (X_i, u_i)_{i \in N}$ is better-reply secure if whenever $(x^*, u^*) \in \text{clGr}G$ and $x^* \in X \setminus E_G$, some player i can secure a payoff strictly above u_i^* at x^* .*

Theorem 1 (Reny 1999) *If $G = (X_i, u_i)_{i \in N}$ is compact, quasiconcave, and better-reply secure, then it possesses a pure strategy Nash equilibrium.*

One can verify that every better-reply secure game possesses the following property (see Reny 2009; Nessah and Tian 2010; or Lemma 7 below).

Definition 3 *A game $G = (X_i, u_i)_{i \in N}$ has the single deviation property if whenever $x \in X \setminus E_G$, there exist $d \in X$ and a neighborhood $U_X(x)$ of x in X such that, for all $x' \in U_X(x)$, there is a player i for whom $u_i(d_i, x'_{-i}) > u_i(x')$.*

We amend the definition of the single deviation property in two important respects. First, the requirement that a deviation strategy, d_i , be defined for each player i can be prohibitive in applications (see the proof of Theorem 3). Second, we ought not to require that there be a player able to increase her payoff for those $x' \in U_X(x)$ that are Nash equilibria of G .

Definition 4 *A game $G = (X_i, u_i)_{i \in N}$ has the weak single deviation property if whenever $x \in X \setminus E_G$, there exist an open neighborhood $U_X(x)$ of x , a set of players $I(x) \subset N$, a collection of points $\{d_i(x) \in X_i : i \in I(x)\}$ such that, for every $x' \in U_X(x) \setminus E_G$, there exists $i \in I(x)$ with $u_i(d_i(x), x'_{-i}) > u_i(x')$.*

To simplify notation, we will write d_i instead of $d_i(x)$ if it is clear for which neighborhood $U_X(x)$ player i 's deviation strategy d_i is used.

Example 1 Consider a two-bidder second-price sealed-bid auction with complete information. Let $0 \leq v_2 < v_1 \leq 1$ and $X_1 = X_2 = [0, 1]$. Bidder i 's payoff function is given by

$$u_i(x_1, x_2) = \begin{cases} 0, & \text{if } x_i < \max\{x_1, x_2\}, \\ \frac{1}{2}(v_i - x_i), & \text{if } x_1 = x_2, \\ v_i - x_{-i}, & \text{if } x_i > x_{-i}, \end{cases}$$

where x_i is player i 's bid and v_i is her valuation of the object. The game possesses the weak single deviation property. To verify this, for every $x \in X \setminus E_G$ choose an open neighborhood $U_X(x)$ that contains x , and put $I(x) = \{1, 2\}$ and $(d_1(x), d_2(x)) = (v_1, v_2)$. At the same time, the game does not have the single deviation property since the set of its pure strategy Nash equilibria is not closed.

Theorem 2 is an equilibrium existence result for games with the weak single deviation property. Its proof is postponed until Section 4, where a more general result, Theorem 5, is shown for qualitative games. For a set A , let $\langle A \rangle$ denote the family of its nonempty finite subsets. In assumption (ii) of Theorem 2, we assume that $d_i(x) = \{\emptyset\}$ if $i \in N \setminus I(x)$.

Theorem 2 *Let $G = (X_i, u_i)_{i \in N}$ be a compact game. Suppose that*

- (i) G has the weak single deviation property, i.e. for each $x \in X \setminus E_G$, there exist an open neighborhood $U_X(x)$ of x , a set of players $I(x) \subset N$, a collection of points $\{d_i(x) \in X_i : i \in I(x)\}$ such that, for every $x' \in U_X(x) \setminus E_G$, there exists $i \in I(x)$ with $u_i(d_i(x), x'_{-i}) > u_i(x')$;*
- (ii) for each $A \in \langle X \setminus E_G \rangle$ and every $z \in \cap_{x \in A} U_X(x)$, there exists $i \in \cup_{x \in A} I(x)$ such that $z_i \notin \text{co}\{\cup_{x \in A} d_i(x)\}$.*

Then G has a pure strategy Nash equilibrium.

It is not difficult to see that assumption (ii) of Theorem 2 is naturally satisfied in a number of games. Among those are both quasiconcave games, such as second-price sealed-bid auctions, and non-quasiconcave games, such as the duopoly game described in Example 1 of Baye, Tian, and Zhou (1993).

A compact, quasiconcave game with the weak single deviation property need not have a pure strategy Nash equilibrium. The next example is borrowed from Reny (2009).

Example 2 Consider a three-player game $G = (X_i, u_i)_{i=1}^3$ with $X_1 = X_2 = X_3 = [0, 1]$. The payoff functions are defined as follows. Let, for $r \in [0, 1]$,

$$u_0(r) = \begin{cases} 0, & \text{if } r > 0, \\ 1, & \text{if } r = 0, \end{cases}, \quad u_1(r) = \begin{cases} 0, & \text{if } r < 1, \\ 1, & \text{if } r = 1. \end{cases}$$

Then, for $x_3 \in [0, \frac{1}{2}]$,

	$x_2 \in [0, \frac{1}{3}]$	$x_2 \in (\frac{1}{3}, \frac{2}{3})$	$x_2 \in [\frac{2}{3}, 1]$
$x_1 \in [0, \frac{1}{2}]$	$(u_0(x_1), u_1(x_2), u_0(x_3))$	$(u_1(x_1), u_1(x_2), u_0(x_3))$	$(u_1(x_1), u_1(x_2), u_1(x_3))$
$x_1 \in (\frac{1}{2}, 1]$	$(u_0(x_1), u_1(x_2), u_0(x_3))$	$(u_1(x_1), u_1(x_2), u_1(x_3))$	$(u_1(x_1), u_1(x_2), u_1(x_3))$

and, for $x_3 \in (\frac{1}{2}, 1]$,

	$x_2 \in [0, \frac{1}{3}]$	$x_2 \in (\frac{1}{3}, \frac{2}{3})$	$x_2 \in [\frac{2}{3}, 1]$
$x_1 \in [0, \frac{1}{2}]$	$(u_0(x_1), u_0(x_2), u_0(x_3))$	$(u_0(x_1), u_0(x_2), u_0(x_3))$	$(u_1(x_1), u_0(x_2), u_1(x_3))$
$x_1 \in (\frac{1}{2}, 1]$	$(u_0(x_1), u_0(x_2), u_0(x_3))$	$(u_0(x_1), u_0(x_2), u_1(x_3))$	$(u_1(x_1), u_0(x_2), u_1(x_3))$

where the first coordinate of each entry corresponds to player 1, the second coordinate to player 2, and the third to player 3. It is not difficult to see that the game is compact, quasiconcave, and has the weak single deviation property. At the same time, it has no Nash equilibrium in pure strategies. Therefore it is impossible to find, for each $x \in X \setminus E_G$, an open neighborhood $U_X(x)$, a set of players $I(x) \subset N$, and a collection of deviation strategies $\{d_i(x) \in X_i : i \in I(x)\}$ satisfying the conditions of Theorem 2.

For compact, quasiconcave, two-player games with the weak single deviation property, the conditions of Theorem 2 can be satisfied if the players' strategy sets are subsets of the real line.

Theorem 3 *If a two-player, compact, quasiconcave game $G = (X_i, u_i)_{i=1}^2$ has the weak single deviation property and each X_i is a subset of the real line \mathbb{R} , then G has a pure strategy Nash equilibrium.*

The proof of Theorem 3 is given in Appendix A. Intuitively, its first step is clear. It proceeds by contradiction. Assume that the game has no pure strategy Nash equilibrium. Then, since the game has the weak single deviation property, we consider a cover of X consisting of open neighborhoods $U_X(x)$, with the corresponding sets of players $I(x)$ and collections of points $\{d_i(x) \in X_i : i \in I(x)\}$. The compactness of X implies that the cover has a finite subcover, and it is tempting to conclude that what is left is to apply Theorem 2. However, Example 3 demonstrates that it is not so.

Example 3 Consider a two-player game $G = (X_i, u_i)_{i=1}^2$ with $X_1 = X_2 = [0, 1]$, and $u_1 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ defined by

$$u_1(x) = \begin{cases} 1 & \text{if } x \in \{\frac{1}{2}\} \times [0, \frac{1}{2}) \text{ and } x \in (0, \frac{1}{2}] \times [\frac{1}{2}, 1], \\ 2 & \text{if } x \in \{0\} \times [\frac{1}{2}, 1], \\ 0 & \text{otherwise,} \end{cases}$$

and $u_2 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ defined by

$$u_2(x) = \begin{cases} 1 & \text{if } x \in (\frac{1}{2}, 1] \times \{\frac{1}{2}\} \text{ and } x \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1), \\ 2 & \text{if } x \in [0, \frac{1}{2}] \times \{1\}, \\ 0 & \text{otherwise.} \end{cases}$$

This game is compact and quasiconcave and has the weak single deviation property.

Consider $U_X(\frac{1}{2}, \frac{1}{2}) = B_X((\frac{1}{2}, \frac{1}{2}), \frac{1}{10})$. For this open neighborhood of $x = (\frac{1}{2}, \frac{1}{2})$, $I(x) = \{1, 2\}$, and, unfortunately, there are two possible ways of choosing the set $\{d_i \in X_i : i \in I(x)\}$, namely $(d_1^1, d_2^1) = (\frac{1}{2}, 1)$ and $(d_1^2, d_2^2) = (0, \frac{1}{2})$. Since $x_i \in \text{co}\{d_i^1, d_i^2\}$ for $i = 1, 2$, it is quite possible that assumption (ii) of Theorem 2 does not hold for a finite subcover consisting of open balls, irrespective of how small its

elements are. As a result, we have to amend the finite cover to make Theorem 2 applicable (see Appendix A for details).

3 The Strong Single Deviation Property

In this section, we introduce a strengthening of the weak single deviation property and show that it is another, slightly weakened, form of the better-reply security condition.

Definition 5 *A game $G = (X_i, u_i)_{i \in N}$ has the strong single deviation property if whenever $x \in X \setminus E_G$, there exist an open neighborhood $U_X(x)$ of x , a set of players $I(x) \subset N$, a family of open neighborhoods $\{U_{X_{-i}}(x_{-i}) : i \in I(x)\}$, a collection of deviation strategies $\{d_i(x) \in X_i : i \in I(x)\}$, and a number $\varepsilon(x) > 0$ such that, for every $x' \in U(x) \setminus E_G$, there exists $i \in I(x)$ with $u_i(d_i(x), z_{-i}) - \varepsilon(x) > u_i(x')$ for all $z_{-i} \in U_{X_{-i}}(x_{-i})$ such that $(d_i(x), z_{-i}) \in X \setminus E_G$.*

A game with the strong single deviation property need not be better-reply secure in the sense of Definition 2.

Example 4 Consider a timing game between two players with $X_1 = X_2 = [0, 1]$. Player i 's payoff function is given by

$$u_i(x_i, x_{-i}) = \begin{cases} 1 & \text{if } x_i < x_{-i}, \\ \varphi_i(x_i) & \text{if } x_i = x_{-i}, \\ -1 & \text{if } x_i > x_{-i}, \end{cases}$$

where $\varphi_i(x_i) = 1$ if $x_i = x_{-i}$ and $x_i < 0.5$, and $\varphi_i(x_i) = 0$ if $x_i = x_{-i}$ and $x_i \geq 0.5$. The set of pure strategy Nash equilibria of this game is $E_G = \{x \in [0, \frac{1}{2}) \times [0, \frac{1}{2}) : x_1 = x_2\}$. It is easy to see that the game is not better-reply secure at $(\frac{1}{2}, \frac{1}{2})$. Now we show that it has the strong single deviation property.

We have to consider two possible cases.

Case 1. If $x \in X$ is such that $x_{-i} < x_i$ for some $i \in \{1, 2\}$, then put $r = \frac{x_i - x_{-i}}{2}$, $I(x) = \{i\}$, $d_i(x) = 0$, $U_X(x) = B_X(x, r)$, $U_{X_{-i}}(x_{-i}) = B_{X_{-i}}(x_{-i}, r)$, and $\varepsilon(x) = 1$.

Case 2. If $x \in X$ is such that $x_1 = x_2$ and $x_1 \geq \frac{1}{2}$, then put $r = \frac{x_1}{2}$, $I(x) = \{1, 2\}$, $(d_1(x), d_2(x)) = (0, 0)$, $U_X(x) = B_X(x, r)$, $U_{X_{-i}}(x_{-i}) = B_{-i}(x_{-i}, r)$, and $\varepsilon(x) = \frac{1}{2}$.

Verifying that the game possesses the strong single deviation property is a straightforward exercise in both cases.

The difference between the strong single deviation property and the better-reply security condition is in the way Nash equilibria are treated.

Lemma 1 *If a compact game $G = (X_i, u_i)_{i \in N}$ with no Nash equilibrium in pure strategies has the strong single deviation property, then it is better-reply secure.*

Proof. Assume that $x^* \in X$ and $u^* \in \mathbb{R}^n$ are such that $(x^*, u^*) \in \text{clGr}G$. Since the game has the strong single deviation property, there exist an open neighborhood $U_X(x^*)$ of x^* , a set of players $I(x^*) \subset N$, a family of open neighborhoods $\{U_{X_{-i}}(x_{-i}^*) : i \in I(x^*)\}$, a collection of deviation strategies $\{d_i \in X_i : i \in I(x^*)\}$, and a number $\varepsilon(x^*) > 0$ such that, for every $x' \in U_X(x^*)$, there exists $i \in I(x)$ with $u_i(d_i, z_{-i}) - \varepsilon(x^*) > u_i(x')$ for all $z_{-i} \in U_{X_{-i}}(x_{-i}^*)$.

We shall show that some player i can secure a payoff strictly above u_i^* at x^* . Consider a net $\{x^\beta\}$ converging to x^* such that the corresponding net $\{u(x^\beta)\}$ tends to u^* . Then there exists $\widehat{\beta}$ such that $x^\beta \in U_X(x^*)$ and $|u_i^* - u_i(x^\beta)| < \frac{\varepsilon(x^*)}{2}$ for all $\beta \succeq \widehat{\beta}$ and all $i \in I(x^*)$. In particular, by the strong single deviation property, the inclusion $x^\beta \in U_X(x^*)$ implies that there exists $i \in I(x^*)$ such that $u_i(d_i, z_{-i}) - \varepsilon(x^*) > u_i(x^\beta)$ for all $z_{-i} \in U_{X_{-i}}(x_{-i}^*)$. Therefore, $u_i(d_i, z_{-i}) - \frac{\varepsilon(x^*)}{2} > u_i^*$ for all $z_{-i} \in U_{X_{-i}}(x_{-i}^*)$, which means that player i can secure $u_i^* + \frac{\varepsilon(x^*)}{2}$ at x^* . ■

The following corollary follows from Theorem 1 and Lemma 1.

Corollary 1 *If $G = (X_i, u_i)_{i \in N}$ is compact, quasiconcave, and has the strong single deviation property, then it possesses a pure strategy Nash equilibrium.*

Remark 1 It is not difficult to relax the better-reply security condition to cover, for example, the above timing game. Instead of considering the graph of G , we can introduce the "nonequilibrium" graph of G by $\text{Grn}G = \{(x, u) \in X \times \mathbb{R}^n \mid u_i(x) = u_i$

for all $i \in N$ and $x \in X \setminus E_G\}$ and replace the set $\text{clGr}G$ in Definition 2 with $\text{clGrn}G$, which will expand the scope of applications of Theorem 1.

The next lemma, along with Lemma 1, shows that the strong single deviation property is another, slightly weakened, form of the better-reply security condition.

Lemma 2 *If a compact game $G = (X_i, u_i)_{i \in N}$ is better-reply secure, then it has the strong single deviation property.*

The proof of Lemma 2 is given in Appendix B.

The closest to the strong single deviation property is the lower single deviation property, introduced by Reny (2009). Its definition is as follows. For each $i \in N$, let $\underline{u}_i : X \rightarrow \mathbb{R}$ be defined by $\underline{u}_i(x_i, x_{-i}) = \liminf_{x'_{-i} \rightarrow x_{-i}} u_i(x_i, x'_{-i})$. A game $G = (X_i, u_i)_{i \in N}$ has the lower single-deviation property if whenever $x \in X \setminus E(G)$, there exists $d \in X$ and a neighborhood U of x such that for all $x' \in U$, there is a player i for whom $\underline{u}_i(d_i, y_{-i}) > \underline{u}_i(x')$ for all $y \in U$.

Since $\varepsilon(x)$ in Definition 5 does not depend on x' and $u_i(x') \geq \underline{u}_i(x')$ for each $i \in N$ and every $x' \in X$, a game with no Nash equilibrium in pure strategies that has the strong single deviation property also has the lower single deviation property. The latter property is a generalization of the better-reply security condition. In its turn, the strong single deviation property may be considered as a slightly improved version of the better-reply security condition. The lower single deviation property can also be improved upon in a similar manner.

4 Equilibrium Existence in Qualitative Games

The proof of the main result of the section, Theorem 5, relies on a generalization of the notion of an L -majorized correspondence (Yannelis and Prabhakar, 1983).

4.1 Domain L -Majorized Correspondences

Let X be a nonempty subset of a topological space, Y be nonempty, convex subset of a vector space, and $\theta : X \rightarrow Y$ be a single-valued function. A correspondence

$F : X \rightrightarrows Y$ has open lower sections in X if $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open in X for every $y \in Y$; F is of class L_θ with respect to θ if $\theta(x) \notin \text{co}F(x)$ for all $x \in X$ and it has open lower sections in X . In the special cases when $Y = X$ and θ is the identity map on X and when $X = \Pi_{i=1}^n X_i$ and $\theta : X \rightarrow X_i$ is the projection of X onto X_i and $Y = X_i$, we will write L in place of L_θ . The domain of F is defined by $\text{Dom}F = \{x \in X : F(x) \neq \emptyset\}$. If $\text{Dom}F = X$, then we say that F is strict.

Given $F : X \rightrightarrows Y$, $\theta : X \rightarrow Y$, and $x \in X$, a correspondence $F_x : X \rightrightarrows Y$ is an L_θ -majorant of F at x if F_x is of class L_θ and there exists an open neighborhood U_x of x in X such that $F(z) \subset F_x(z)$ for every $z \in U_x$.³ The correspondence F is locally L_θ -majorized if, for each $x \in \text{Dom}F$, there exists an L_θ -majorant of F at x ; and F is L_θ -majorized if there exists a correspondence $\bar{F} : X \rightrightarrows Y$ of class L_θ such that $F(x) \subset \bar{F}(x)$ for every $x \in X$.

The next maximal element existence result is equivalent to Browder's fixed point theorem (see Browder 1968, Theorem 1; Yannelis and Prabhakar 1983, Theorems 3.1 and 5.1).

Lemma 3 *Let X be a nonempty, compact, convex subset of a Hausdorff topological vector space, and let $F : X \rightrightarrows X$ be a correspondence of class L . Then there exists $\hat{x} \in X$ such that $F(\hat{x}) = \emptyset$.*

The following lemma shows that, from the standpoint of applications, there are no differences between L_θ -majorized and locally L_θ -majorized correspondences (see Yannelis and Prabhakar 1983, Corollary 5.1; or Bagh 1998, Lemma 1.5).

Lemma 4 *Let X be a nonempty, compact subset of a Hausdorff topological vector space, Y be a nonempty, convex subset of a vector space, and $\theta : X \rightarrow Y$. Let $F : X \rightrightarrows Y$ be a strict correspondence. Then F is locally L_θ -majorized if and only if it is L_θ -majorized.*

Among the conditions of Lemma 4 is a nonstandard one, namely that F is a strict correspondence. This condition is not restrictive since every proof using majorization tools proceeds by contradiction.

³In this section, we will write U_x instead of $U_X(x)$ since there is no ambiguity regarding the space in which the neighborhood is considered.

The proof of Lemma 4 follows along the lines of the proof of Corollary 1 of Borglin and Keiding (1976) (for details, see Ding et al. 1994, Theorem 1; Ding and Tan 1993, Lemma 2, for the case when X is a paracompact topological space).

Corollary 2 (Yannelis and Prabhakar 1983, Corollary 5.1) follows from Lemmas 3 and 4 by way of contradiction.

Corollary 2 *Let X be a nonempty, compact, convex subset of a Hausdorff topological vector space and $F : X \rightrightarrows X$ be a locally L -majorized correspondence. Then there exists $\hat{x} \in X$ such that $F(\hat{x}) = \emptyset$.*

Now, we introduce a generalization of the notion of an L -majorized correspondence.

Definition 6 *Let X be a nonempty subset of a topological space, Y be a nonempty, convex subset of a vector space, and let $\theta : X \rightarrow Y$. A correspondence $F : X \rightrightarrows Y$ is domain L_θ -majorized if there exists a correspondence $\bar{F} : X \rightrightarrows Y$ of class L_θ such that $\text{Dom}F \subset \text{Dom}\bar{F}$.*

Clearly, if F is L_θ -majorized, then it is domain L_θ -majorized. Obviously, the converse does not necessarily hold.

Theorem 4 is a maximal element existence theorem for domain L -majorized correspondences.

Theorem 4 *Let X be a nonempty, compact, convex subset of a Hausdorff topological vector space and $F : X \rightrightarrows X$ be a domain L -majorized correspondence. Then there exists $\hat{x} \in X$ such that $F(\hat{x}) = \emptyset$.*

Proof. Since $F : X \rightrightarrows X$ is domain L -majorized, there is a correspondence $\bar{F} : X \rightrightarrows X$ of class L such that $\text{Dom}F \subset \text{Dom}\bar{F}$. Then, by Lemma 3, $\bar{F}(\hat{x}) = \emptyset$ for some $\hat{x} \in X$, which implies that $F(\hat{x}) = \emptyset$. ■

The next lemma provides a set of sufficient conditions for a correspondence to be domain L_θ -majorized.

Lemma 5 *Let X be a compact Hausdorff topological space and Y be a nonempty, convex subset of a vector space. Let $\theta : X \rightarrow Y$ and $F : X \rightrightarrows Y$ be a strict correspondence such that*

- (i) for each $x \in X$, there exist an open neighborhood U_x of x in X and a correspondence $F_x : X \rightrightarrows Y$ with $\text{Dom} F_x = U_x$ and open lower sections in X ;*
- (ii) for each $A \in \langle X \rangle$ and every $z \in \cap_{x \in A} U_x$, $\theta(z) \notin \text{co}\{\cup_{x \in A} F_x(z)\}$.*

Then F is domain L_θ -majorized.

The proof of Lemma 5 is given in Appendix B, where is shown that assumptions (i) and (ii) imply the existence of a strict correspondence $\bar{F} : X \rightrightarrows Y$ of class L_θ .

It is worth noticing that another set of sufficient conditions obtains if assumption (ii) is replaced with the more conventional assumption that $\cap_{x \in A} U_x \subset \text{Dom}(\cap_{x \in A} F_x)$ for each $A \in \langle X \rangle$ (see, e.g., Yuan 1999, Theorem 3.1). However, the latter assumption has a strong flavor of value majorization.

4.2 Qualitative Games

As before, let $N = \{1, \dots, n\}$ be a finite set of players. Each player i 's strategy set X_i is a nonempty, compact, and convex subset of a Hausdorff topological vector space. Let $X = \prod_{i \in N} X_i$ and $P_i : X \rightrightarrows X_i$ be player i 's preference correspondence.

Consider a qualitative game $\Gamma = (X_i, P_i)_{i \in N}$. A strategy profile $x \in X$ is an equilibrium of Γ if $P_i(x) = \emptyset$ for all $i \in N$.

For a qualitative game $\Gamma = (X_i, P_i)_{i \in N}$, we call the set $\text{Dom} \Gamma = \cup_{i \in N} \text{Dom} P_i$ the domain of Γ .

Definition 7 *A game $\Gamma = (X_i, P_i)_{i \in N}$ is domain L -majorized if there exists a correspondence $\bar{F} : X \rightrightarrows X$ of class L such that $\text{Dom} \Gamma \subset \text{Dom} \bar{F}$.*

Obviously, if Γ is domain L -majorized, then it has an equilibrium $\hat{x} \in X$; that is, $P_i(\hat{x}) = \emptyset$ for each $i \in N$. Therefore, if we want to show the existence of an equilibrium in a qualitative game Γ , a legitimate way of doing that is to show that the game is domain L -majorized. However, it is important to keep in mind that

the correspondence \overline{F} should not only have open lower sections but also satisfy the condition that $x \notin \text{co}\overline{F}(x)$ for all $x \in X$.

Extending Lemma 5 to qualitative games produces an equilibrium existence result which is analogous to Theorem 2.2 of Barelli and Soza (2009).

Corollary 3 *Let X_i be a nonempty, compact, convex subset of a Hausdorff topological vector space and $\Gamma = (X_i, P_i)_{i \in N}$ be a qualitative game. Suppose, for each $x \in \text{Dom}\Gamma$, there exist an $(n+1)$ -tuple $(D_x^1, \dots, D_x^n; U_x)$, where $D_x^i : X \rightrightarrows X_i$ and U_x is an open neighborhood of x in X , such that*

- (i) $\text{Dom}D_x^i = U_x$ and D_x^i has open lower sections in X for all $i \in N$;
- (ii) for each $A \in \langle \text{Dom}\Gamma \rangle$ and every $z \in \cap_{x \in A} U_x$, there exists $i \in N$ such that $z_i \notin \text{co}\{\cup_{x \in A} D_x^i(z)\}$.

Then Γ has an equilibrium.

Proof. Assume, by contradiction, that Γ has no equilibrium. For each $x \in X$, consider $F_x : X \rightrightarrows X$ defined by $F_x(z) = (D_x^1(z), \dots, D_x^n(z))$. Since assumptions (i) and (ii) of Lemma 5 are satisfied, Γ is domain L -majorized, a contradiction. ■

From the standpoint of applications, assumption (i) of Corollary 3 is too strong. Intuitively, if $P_i(x) = \emptyset$ for some i , then $\text{Dom}D_x^i$ should be the empty set as well. Moreover, assuming that (i) holds for all $i \in N$ makes it more difficult, or even in some cases impossible, to satisfy (ii).

Theorem 5 *Let X_i be a nonempty, compact, convex subset of a Hausdorff topological vector space and $\Gamma = (X_i, P_i)_{i \in N}$ be a qualitative game. Suppose, for each $x \in \text{Dom}\Gamma$, there exist $I(x) \subset N$, and an $(n+1)$ -tuple $(D_x^1, \dots, D_x^n; U_x)$, where $D_x^i : X \rightrightarrows X_i$ and U_x is an open neighborhood of x in X , such that*

- (i) $\text{Dom}D_x^i = U_x$ and D_x^i has open lower sections in X for all $i \in I(x)$, and $\text{Dom}D_x^i = \emptyset$ for all $i \in N \setminus \{I(x)\}$;
- (ii) for each $A \in \langle \text{Dom}\Gamma \rangle$ and every $z \in \cap_{x \in A} U_x$, there exists $i \in \cup_{x \in A} I(x)$ such that $z_i \notin \text{co}\{\cup_{x \in A} D_x^i(z)\}$.

Then Γ has an equilibrium.

Proof. Assume, by contradiction, that Γ has no equilibrium, i.e. $\text{Dom}\Gamma = X$. The compactness of X implies that the open covering $\{U_x : x \in X\}$ of X contains a finite subcover $\{U_{x_j} : j \in J\}$, where J is a finite set. Let $\{V_{x_j} : j \in J\}$ be an open refinement of $\{U_{x_j} : j \in J\}$ such that $\text{cl}V_{x_j} \subset U_{x_j}$ for every $j \in J$ (see Aliprantis and Border 2006, pp. 169-171). For each $j \in J$ and each $i \in N$, define a correspondence $F_i^j : X \rightrightarrows X_i$ by

$$F_i^j(z) = \begin{cases} D_{x_j}^i(z) \cup_{\{s \in J \setminus \{j\} : z \in V_{x_s}\}} D_{x_s}^i(z) & \text{if } z \in \text{cl}V_{x_j} \text{ and } i \in I(x_j), \\ X_i & \text{if } z \notin \text{cl}V_{x_j} \text{ or } i \notin I(x_j). \end{cases}$$

It is not difficult to see that each F_i^j has open lower sections. Then for each $i \in N$, the correspondence $F_i : X \rightrightarrows X_i$ defined by $F_i(z) = \cap_{j \in J} F_i^j(z)$ has open lower sections. Therefore, the correspondence $\bar{F} : X \rightrightarrows X$ defined by $\bar{F}(z) = \cap_{i \in N} \{\prod_{k \in N \setminus \{i\}} X_k \times F_i(z)\}$ also has open lower sections.

Fix some $z \in X$. It lies in some V_{x_j} . We have to show that $z_{i'} \notin \text{co}F_{i'}^j(z)$ for some $i' \in I(x_j)$. Denote $A = \{s \in J : z \in V_{x_s}\}$. Then assumption (ii) implies that there exists $i' \in \cup_{x \in A} I(x)$ such that $z_{i'} \notin \text{co}\{\cup_{j \in A} D_{x_j}^{i'}(z)\}$. Since, by definition, $F_{i'}^j(z) = \cup_{j \in A} D_{x_j}^{i'}(z)$, we conclude that $z_{i'} \notin \text{co}F_{i'}^j(z)$. Therefore $z_{i'} \notin \text{co}F_{i'}(z)$, and, consequently, $z \notin \text{co}\bar{F}(z)$.

Since \bar{F} is of class L and $\text{Dom}\Gamma = \text{Dom}\bar{F} = X$, Γ is domain L -majorized, a contradiction. ■

Corollary 4 is a version of the Fan-Browder collective fixed point theorem. In Lassonde and Schenkel (1992, Theorem 5), it follows from a generalization of the KKM lemma, which is a reflection of the fact that the KKM lemma and Browder's fixed point theorem are two equivalent results (see Yannelis 1991, pp. 105-109, for an in-depth explanation).⁴

Corollary 4 *Let X_1, \dots, X_n be nonempty, compact, convex subsets of Hausdorff topological vector spaces, and $X = \prod_{i \in N} X_i$. For each $i \in N$, let $D_i : X \rightrightarrows X_i$ have*

⁴Prokopovych (2011) uses the Fan-Browder collective fixed point theorem in its classical form to show Reny's equilibrium existence theorem.

open lower sections. If for each $x \in X$, there exists $i \in N$ such that $D_i(x) \neq \emptyset$, then there exists $\bar{x} \in X$ and $i \in N$ such that $\bar{x} \in \text{co}D_i(\bar{x})$.

Proof. Assume, by contradiction, that each correspondence D_i is of class L . Consider the game $\Gamma = (X_i, D_i)_{i \in N}$. For each $x \in X$, put $I(x) = \{i \in N : D_i(x) \neq \emptyset\}$ and fix a neighborhood U_x such that $U_x \subset \text{Dom}D_i$ for all $i \in I(x)$. Then define $D_i^x : X \rightarrow X_i$ as a restriction of D_i to U_x for $i \in I(x)$ and put $\text{Dom}D_i^x = \emptyset$ for $i \in N \setminus I(x)$. By Theorem 5, Γ has an equilibrium, a contradiction. ■

Corollary 5 is a strengthening of Corollary 5 for qualitative games.

Corollary 5 *Let each X_i be a nonempty, compact, convex subset of a Hausdorff topological vector space and let $\Gamma = (X_i, P_i)_{i \in N}$ be a qualitative game. Assume that, for each $i \in N$, the correspondence $P_i : X \rightarrow X_i$ is domain L -majorized. Then Γ has an equilibrium in X .*

Proof. Since the players' preference correspondences are domain L -majorized, for each $i \in N$ there exists $F_i : X \rightarrow X_i$ of class L such that $\text{Dom}P_i \subset \text{Dom}F_i$.

By Corollary 4, there exists $x \in X$ such that $F_i(x) = \emptyset$ for all $i \in N$. Since $\text{Dom}\Gamma = \cup_{i \in N} \text{Dom}P_i \subset \cup_{i \in N} \text{Dom}F_i$, Γ has an equilibrium. ■

Appendix A

This appendix contains the proof of Theorem 3 and an intuitive explanation of the amending technique used in the proof.

Proof of Theorem 3

Assume, by contradiction, that G has no Nash equilibrium in pure strategies. Since G has the weak single deviation property, for every $x \in X$ there exist an open ball $B_X(x, 3r(x))$ of x in X , a set of players $I(x) \subset \{1, 2\}$, and a collection of deviation strategies $K(x) = \{d_i \in X_i : i \in I(x)\}$ such that, for every $x' \in B_X(x, 3r(x))$, $u_i(d_i, x'_{-i}) > u_i(x')$ for some $i \in I(x)$.

We will amend the initial open cover of X , $\{B_X(x, r(x)) : x \in X\}$, in a number of steps. First, we additionally assume that, for every $x \in X$, $r(x)$, $I(x)$, and $K(x)$ satisfy the following three conditions:

- (a) if $d_i \neq x_i$ for some $i \in I(x)$, then $|d_i - x_i| > 5r(x)$;
- (b) $I(x)$ is minimal in the following sense: If $I = \{1, 2\}$, there are no $r > 0$ and $i \in \{1, 2\}$ such that $u_i(d_i, x'_{-i}) > u_i(x')$ for all $x' \in B_X(x, r)$;
- (c) if $d_i = x_i$ for some $i \in I(x)$, then d_i can not be replaced in $K(x)$ with $\bar{d}_i \in X_i \setminus \{d_i\}$ such that, for some $r > 0$ and every x' in $B_X(x, 3r)$, at least one of the following inequalities holds: $u_i(\bar{d}_i, x'_{-i}) > u_i(x')$ or $u_{-i}(d_{-i}, x'_i) > u_{-i}(x')$.

As is shown below, if the elements of the cover satisfy these three simple conditions, we do not have to further amend them in most cases. Condition (a) is not restrictive since, for every $x \in X$, the radius $r(x)$ can be chosen arbitrarily small.

If $I(x) = \{1, 2\}$, then condition (b) states that, given $K(x)$, it is impossible to reduce the number of elements of $I(x)$ by choosing a smaller $r(x)$. One can notice that condition (c) is also not burdensome. If $d_i = x_i$ for some $i \in I(x)$ and there are $r > 0$ and $\bar{d}_i \in X_i \setminus \{d_i\}$ such that $u_i(\bar{d}_i, x'_{-i}) > u_i(x')$ for every $x' \in B_X(x, 3r)$ at which $u_i(d_i, x'_{-i}) > u_i(x')$ and $u_{-i}(d_{-i}, x'_i) \leq u_{-i}(x')$, then we replace $B_X(x, r(x))$ in the cover with $B_X(x, r)$ and d_i in $K(x) = \{d_1, d_2\}$ with \bar{d}_i . A useful fact to keep in mind is that if $d_i = x_i$ for some $i \in I(x)$, then d_{-i} does not coincide with x_{-i} .

The compactness of X implies that the open cover $\{B_X(x, r(x)) : x \in X\}$ of X contains a finite subcover $\{B_X(x^j, r(x^j)) : j \in J\}$, where $J = \{1, \dots, k\}$. It is useful to notice that if $B_X(x^s, r(x^s)) \cap B_X(x^t, r(x^t)) \neq \emptyset$ and $r(x^s) > r(x^t)$ for some $s, t \in J$, then $B_X(x^t, r(x^t)) \subset B_X(x^s, 3r(x^s))$. Hence for every $x' \in B_X(x^t, r(x^t))$, there exist $i \in I(x^s)$ and $d_i^s \in K(x^s)$ such that $u_i(d_i^s, x'_{-i}) > u_i(x')$.

Without loss of generality, we also assume that $r(x^s) > r(x^t)$ if $s, t \in J$ and $s < t$, and that each $B_X(x^j, r(x^j))$ contains some points of X that do not lie in any of the other elements of the subcover. The latter assumption will help us avoid dealing with empty sets in the course of amending the cover.

Let us show that, for our purposes, it is enough to focus attention on the intersections of just two elements of the cover. In the reasoning below, the fact that the open sets are open balls is not essential. So, we want to show that if

for some $\{l_1, \dots, l_m\} \subset \{2, \dots, k\}$, there exists $z' \in \cap_{j=1}^m B_X(x^{l_j}, r(x^{l_j}))$ such that $z'_i \in \text{co}\{\cup_{j=1}^m d_i^{l_j}\}$ for $i = 1, 2$ (here we assume that $d_i^{l_j} = \{\emptyset\}$ for $i \in \{1, 2\} \setminus I(x^{l_j})$), then $z'_i \in \text{co}\{d_i^s, d_i^t\}$, $i = 1, 2$, for some $s, t \in \{l_1, \dots, l_m\}$.

Without loss of generality, $\{l_1, \dots, l_m\} = \{1, \dots, m\}$, $u_1(d_1^1, z'_2) > u_1(z')$ for $d_1^1 \in K(x^1)$, and $d_1^1 < z'_1$. Since $z'_1 \in \text{co}\{d_1^1, d_1^s\}$ for some $s \in \{2, \dots, m\}$ and u_1 is quasiconcave in x_1 , we have that $z'_1 \leq d_1^s$ and $u_2(z'_1, d_2^s) > u_2(z')$. Then for some $t \in \{1, \dots, m\}$ such that $z'_2 \in \text{co}\{d_2^s, d_2^t\}$, the inequality $u_1(d_1^t, z'_2) > u_1(z')$ holds. It follows from the quasiconcavity of u_1 in x_1 that $d_1^t < z'_1$. That is, $z'_i \in \text{co}\{d_i^s, d_i^t\}$ for $i = 1, 2$, as claimed.

Now consider the intersection of the first two elements of the cover. Let there be a point $z' \in B_X(x^1, r(x^1)) \cap B_X(x^2, r(x^2))$ such that $z'_i \in \text{co}\{d_i^1, d_i^2\}$, $i = 1, 2$. First, we will show that it might happen only in one case (Case 7), and then we will describe how to amend the cover to preclude Case 7 for every pair of intersecting elements of the cover.

As before, we assume that $r(x^1) > r(x^2)$ and that $u_1(d_1^1, z'_2) > u_1(z')$ with $d_1^1 < z'_1$ and $u_2(z'_1, d_2^2) > u_2(z')$ with $d_2^2 > z'_2$ (if it is not so, renumber the players and/or redirect one or both axes). Then the inclusions $z'_i \in \text{co}\{d_i^1, d_i^2\}$ for $i = 1, 2$ imply that $d_1^2 \geq z'_1$ and $d_2^1 \leq z'_2$.

We claim that if $z'_i \in \text{co}\{d_i^1, d_i^2\}$ for $i = 1, 2$, then the following six cases are impossible (see the proof of this claim below).

Case 1. $d_1^1 = x_1^1$ and $d_2^2 = x_2^2$.

Case 2-3. $d_1^1 = x_1^1$ and $d_1^2 \neq x_1^2$.

Therefore, if Cases 1-3 are impossible, then it must be the case that $d_1^1 \neq x_1^1$.

Case 4. $d_1^1 \neq x_1^1$ and $d_1^2 \neq x_1^2$.

Case 5. $d_1^1 \neq x_1^1$, $d_1^2 = x_1^2$, and $d_2^1 \neq x_2^1$.

Cases 4 and 5 complement the picture. We conclude from Case 4 that if $z'_i \in \text{co}\{d_i^1, d_i^2\}$, $i = 1, 2$, then $d_1^2 = x_1^2$, and from Case 5 that, moreover, $d_2^1 = x_2^1$. The next case is also impossible.

Case 6. $d_2^1 = x_2^1$, $d_2^2 = x_2^2$, and $x_2^2 > d_2^1$.

It might happen that $z'_i \in \text{co}\{d_i^1, d_i^2\}$, $i = 1, 2$, in Case 7. As a result, we have to amend the cover so that to ensure that every two of its elements do not satisfy the

conditions of Case 7.

Case 7. Let $d_1^2 = x_1^2$, $d_2^1 = x_2^1$, and $x_2^2 \leq d_2^1$. Denote $B_X(x^1, r(x^1))$ by $V_X^1(x^1, r(x^1))$ and replace $B_X(x^1, r(x^1))$ in the cover $\{B_X(x^j, r(x^j)) : j \in J\}$ with $V_X^2(x^1, r(x^1)) = B_X(x^1, r(x^1)) \setminus \text{cl} B_X(x^2, r(x^2))$. We have to add to the cover a finite number of open balls covering the compact set $A = \partial B_X(x^2, r(x^2)) \cap \text{cl} B_X(x^1, r(x^1))$. For every $x \in A$ with $x_2 \neq d_2^1$, pick an open ball $B_X(x, r(x))$ such that $|x_2 - d_2^1| > 5r(x)$ and find a minimal $I(x) \subset I(x^1)$ (for $B_X(x, r(x))$) with $K(x) \subset K(x^1)$ such that for every $x' \in B_X(x, 3r(x))$, there exists $i \in I(x)$ with $u_i(d_i, x'_{-i}) > u_i(x')$.

Since $x_2^2 \leq d_2^1$ and $z'_2 \geq d_2^1$, we deduce that $(d_1^2, d_2^1) \notin A$. Then for every $x \in A$ with $x_2 = d_2^1$, it is possible to choose an open ball $B_X(x, r(x))$ such that $|x_1 - d_1^2| > 5r(x)$ and find a minimal $I(x) \subset I(x^2)$ with $K(x) \subset K(x^2)$ such that for every $x' \in B_X(x, 3r(x))$, there exists $i \in I(x)$ with $u_i(d_i, x'_{-i}) > u_i(x')$.

Since A is a compact set, it has a finite subset $\{x_A^1, \dots, x_A^T\}$ such that $A \subset \cup_{t=1}^T B_X(x_A^t, r(x_A^t))$. Without loss of generality, $r(x^k) > r(x_A^1)$ and $r(x_A^t) > r(x_A^s)$ if $t, s \in \{1, \dots, T\}$ and $t < s$. Let $x^{k+t} = x_A^t$ for all $t \in \{1, \dots, T\}$. Consider the finite cover of X consisting of $V_X^2(x^1, r(x^1))$, $B_X(x^2, r(x^2))$, $\dots, B_X(x^k, r(x^k))$, $B_X(x^{k+1}, r(x^{k+1}))$, $\dots, B_X(x^{k+T}, r(x^{k+T}))$.

For the open set $V_X^2(x^1, r(x^1))$, we use the same sets $I(x^1)$ and $K(x^1)$ as for $B_X(x^1, r(x^1))$. The balls that we have added to the initial cover satisfy conditions (a)-(c). Moreover, $d_i^{k+t} \neq x_i^{k+t}$ for all $i \in I(x^{k+t})$ and all $t \in \{1, \dots, T\}$. Hence, if a ball added to the cover, for example $B_X(x^s, r(x^s))$, $s \in \{k, \dots, k+T\}$, intersects another element of the cover, denoted by $V(x^j)$, then, for every $z' \in B_X(x^s, r(x^s)) \cap V(x^j)$, we have that $z'_i \notin \text{co}\{d_i^s, d_i^j\}$ for $i = 1, 2$. This is so because $V(x^j)$ is either a subset of or equal to an open ball satisfying conditions (a)-(c), and $d_i^s \neq x_i^s$ for all $i \in I(x^s)$, which precludes Case 7. It is worth mentioning that x^1 need not belong to $V_X^2(x^1, r(x^1))$.

Then consider the sets $V_X^2(x^1, r(x^1))$ and $B_X(x^3, r(x^3))$. If needed, we again amend the cover of X with the help of the just described technique, denoting $V_X^2(x^1, r(x^1)) \setminus \text{cl} B_X(x^3, r(x^3))$ by $V_X^3(x^1, r(x^1))$. Otherwise we put $V_X^3(x^1, r(x^1)) = V_X^2(x^1, r(x^1))$. After considering all the pairs $V_X^{j-1}(x^1, r(x^1))$ and $B_X(x^j, r(x^j))$, $j = 2, \dots, k$, we denote $V(x^1) = V_X^k(x^1, r(x^1))$ and proceed to considering $B_X(x^2, r(x^2))$ and $B_X(x^3, r(x^3))$, and so on. If needed, the amending technique is applied again.

One can see that the last ball that might need amending is $B_X(x^{k-1}, r(x^{k-1}))$. Hence, $V(x^s) = B_X(x^s, r(x^s))$ for $s \geq k$. So, after a finite number of rounds of amendment, we will get a finite open cover of X , $\{V(x^j) : j = 1, \dots, R\}$ with $I(x^j)$ and $\{d_i^j \in X_i : i \in I(x^j)\}$, such that

- (a) for every $x' \in V(x^j)$, $u_i(d_i^j, x'_{-i}) > u_i(x')$ for some $i \in I(x^j)$;
- (b) for every pair $s, t \in \{1, \dots, R\}$, $s \neq t$, if $z' \in V(x^s) \cap V(x^t)$, then $z'_i \notin \text{co}\{d_i^s, d_i^t\}$ for some $i \in \{1, 2\}$, where again we assume that $d_i^j = \{\emptyset\}$ if $i \in \{1, 2\} \setminus I(x^j)$.

Let x be some point of X . Since $\{V(x^1), \dots, V(x^R)\}$ is a cover of X , there exists $V(x^j)$, $j \in \{1, \dots, R\}$, such that $x \in V(x^j)$. Put $U_X(x) = V(x^j)$ and $I(x) = I(x^j)$. For each $i \in I(x)$, set $d_i(x) = d_i^j$. Then the conditions of Theorem 2 are satisfied, a contradiction.

Proofs for Cases 1-6

The following fact will be used frequently below: It follows from the quasiconcavity of $u_1(u_2)$ in $x_1(x_2)$ and the inclusion $z'_1 \in \text{co}\{d_1^1, d_1^2\}$ ($z'_2 \in \text{co}\{d_2^1, d_2^2\}$) that if $u_1(d_1^1, z'_2) > u_1(z')$ ($u_2(z'_1, d_2^2) > u_2(z')$), then $u_1(d_1^1, z'_2) > u_1(d_1^2, z'_2)$ ($u_2(z'_1, d_2^2) > u_2(z'_1, d_2^1)$). So it is important to keep in mind that our assumptions imply that $u_1(d_1^1, z'_2) > u_1(d_1^2, z'_2)$ and $u_2(z'_1, d_2^2) > u_2(z'_1, d_2^1)$.

Case 1. Assume, by contradiction, that $d_1^1 = x_1^1$ and $d_1^2 = x_1^2$. Then $u_2(d_1^2, d_2^2) > u_2(d_1^2, z_2)$ for all $(d_1^2, z_2) \in B_X(x^2, r(x^2))$. Then the containment $B_X(x^2, r(x^2)) \subset B_X(x^1, 3r(x^1))$ and the quasiconcavity of u_2 in x_2 imply that it must be the case that $u_1(d_1^1, z_2) > u_1(d_1^2, z_2)$ for all $(d_1^2, z_2) \in B_X(x^2, r(x^2))$, which contradicts condition (c).

Case 2. We now assume that $d_1^1 = x_1^1$, $d_1^2 \neq x_1^2$, and $(d_1^1, z'_2) \in B_X(x^2, 3r(x^2))$. Since $(d_1^1, z'_2) \in B_X(x^2, 3r(x^2))$ and $u_1(d_1^1, z'_2) > u_1(d_1^2, z'_2)$, we have that $u_2(d_1^1, d_2^2) > u_2(d_1^1, z'_2) \geq u_2(d_1^1, d_2^1)$. However, the inclusion $(d_1^1, z'_2) \in B_X(x^1, r(x^1))$ implies that it must be the case that $u_2(d_1^1, d_2^1) > u_2(d_1^1, z'_2)$, a contradiction.

Case 3. Let $d_1^1 = x_1^1$, $d_1^2 \neq x_1^2$, and $(d_1^1, z'_2) \notin B_X(x^2, 3r(x^2))$. It is worth noticing that if $x_1^2 \leq d_1^1$, then (d_1^1, z'_2) would be located closer to x^2 than z' , which is impossible since $z' \in B_X(x^2, r(x^2))$. Therefore, $x_1^2 > d_1^1$. We have to consider the following two subcases: $d_2^2 = x_2^2$ and $d_2^2 \neq x_2^2$.

Case 3.1. If $d_2^2 = x_2^2$, then $u_1(d_1^2, d_2^2) > u_1(z_1', d_2^2) \geq u_1(d_1^1, d_2^2)$. Since $(z_1', d_2^2) \in B_X(x^1, 3r(x^1))$ and $u_1(z_1', d_2^2) \geq u_1(d_1^1, d_2^2)$, it must be the case that $u_2(z_1', d_2^2) > u_2(z_1', d_2^1)$, a contradiction.

Case 3.2. Let $d_2^2 \neq x_2^2$. First, we claim that $u_2(z_1, d_2^2) > u_2(z_1, z_2') \geq u_2(z_1, d_2^1)$ for all $(z_1, z_2') \in B_X(x^2, r(x^2))$. Assume, to the contrary, that $u_2(z_1, d_2^2) \leq u_2(z_1, z_2')$ for some $(z_1, z_2') \in B_X(x^2, r(x^2))$. Then $u_1(d_1^2, z_2') > u_1(z_1, z_2') \geq u_1(d_1^1, z_2')$, a contradiction.

Now we claim that, in contradiction to condition (b), $u_2(z_1, d_2^2) > u_2(z)$ for all $z \in B_X(x^2, r(x^2))$ with z_1 such that $(z_1, z_2') \in B_X(x^2, r(x^2))$.

Assume, by contradiction, that $u_2(z_1, d_2^2) \leq u_2(z)$ for some $z \in B_X(x^2, r(x^2))$ with $z_2 > z_2'$ and z_1 such that $(z_1, z_2') \in B_X(x^2, r(x^2))$. Then $u_1(d_1^2, z_2) > u_1(z)$. Since $(d_1^1, z_2') \notin B_X(x^2, 3r(x^2))$ and $(z_1, z_2') \in B_X(x^2, r(x^2))$, we have that $d_1^1 < z_1$ and, therefore, $u_1(z) \geq u_1(d_1^1, z_2)$. Then $u_2(z_1, d_2^1) > u_2(z) \geq u_2(z_1, d_2^2)$, a contradiction.

Case 4. Let $d_1^1 \neq x_1^1$ and $d_1^2 \neq x_1^2$. Then $u_1(d_1^1, z_2') > u_1(z_1, z_2')$ for all $(z_1, z_2') \in B_X(x^2, r(x^2))$. Let us show this for the sake of completeness. Assume, by contradiction, that $u_1(d_1^1, z_2') \leq u_1(z_1, z_2')$ for some $(z_1, z_2') \in B_X(x^2, r(x^2))$ with $z_1 < z_1'$. Then $u_2(z_1, d_2^1) > u_2(z_1, z_2')$, and, hence, $u_1(d_1^2, z_2') > u_1(z_1, z_2')$. This implies that $u_1(d_1^2, z_2') > u_1(d_1^1, z_2')$, a contradiction.

Therefore $u_2(z_1, d_2^2) > u_2(z_1, z_2') \geq u_2(z_1, d_2^1)$ for all $(z_1, z_2') \in B_X(x^2, r(x^2))$. We have to consider the following two subcases: 1) $d_2^2 = x_2^2$ and 2) $d_2^2 \neq x_2^2$.

Case 4.1. Let $d_2^2 = x_2^2$. Since $u_1(d_1^2, d_2^2) > u_1(z_1', d_2^2)$, it must be the case that $u_2(z_1', d_2^1) > u_2(z_1', d_2^2)$, a contradiction.

Case 4.2. Let $d_2^2 \neq x_2^2$. Then, in contradiction to the minimality property of $I(x^2)$, $u_2(z_1, d_2^2) > u_2(z)$ for all $z \in B_X(x^2, r(x^2))$ with z_1 such that $(z_1, z_2') \in B_X(x^2, r(x^2))$. Let us show this.

Assume, by contradiction, that $u_2(z_1, d_2^2) \leq u_2(z)$ for some $z \in B_X(x^2, r(x^2))$ with $z_2 > z_2'$ and z_1 such that $(z_1, z_2') \in B_X(x^2, r(x^2))$. Then $u_1(d_1^2, z_2) > u_1(z)$, and, hence, $u_2(z_1, d_2^1) > u_2(z) \geq u_2(z_1, d_2^2)$, a contradiction.

Case 5. Let $d_1^1 \neq x_1^1$, $d_2^1 \neq x_2^1$, and $d_1^2 = x_1^2$. Since $u_2(d_1^2, d_2^2) > u_2(d_1^2, z_2)$ for all $(d_1^2, z_2) \in B_X(x^2, r(x^2))$, we have that $u_1(d_1^1, z_2) > u_1(d_1^2, z_2)$ for all $(d_1^2, z_2) \in$

$B_X(x^2, r(x^2))$. In order to obtain a contradiction with condition (c), it is enough to show that $u_1(d_1^1, z_2) > u_1(z)$ for all $z \in B_X(x^2, r(x^2))$.

Assume that $u_1(d_1^1, z_2) \leq u_1(z)$ for some $z \in B_X(x^2, r(x^2))$ with $z_1 < d_1^1$. Then it must be the case that $u_2(z_1, d_2^1) > u_2(z)$, and, therefore, $u_1(d_1^2, z_2) > u_1(z) \geq u_1(d_1^1, z_2)$, a contradiction.

Case 6. Let $d_2^1 = x_2^1$, $d_1^2 = x_1^2$, and $x_2^2 > d_2^1$. Since $u_2(d_1^2, d_2^2) > u_2(d_1^2, z_2)$ for all $(d_1^2, z_2) \in B_X(x^2, r(x^2))$, we have that $u_1(d_1^1, z_2) > u_1(d_1^2, z_2)$ for all $(d_1^2, z_2) \in B_X(x^2, r(x^2))$ with $z_2 \geq d_2^1$. Then one can show that $u_1(d_1^1, z_2) > u_1(z)$ for all $z \in B_X(x^2, r(x^2))$ with $z_2 \geq d_2^1$, which contradicts condition (c).

The Amending Technique: Example 3 (continued)

Now, using Example 3, we explain the intuition behind the amending technique used in the proof of Theorem 5.

Let $U_X(x^1) = B_X(x^1, r(x^1))$ with $x^1 = (\frac{1}{2}, \frac{1}{2})$, $r(x^1) = \frac{1}{10}$, $I(x^1) = \{1, 2\}$, and $(d_1^1, d_2^1) = (0, \frac{1}{2})$, and $U_X(x^2) = B_X(x^2, r(x^2))$ with $x^2 = (\frac{1}{2}, \frac{5}{12})$, $r(x^2) = \frac{1}{11}$, $I(x^2) = \{1, 2\}$ and $(d_1^2, d_2^2) = (\frac{1}{2}, 1)$. It is not difficult to see that conditions (a)-(c) of the proof of Theorem 3 are satisfied for these two open balls. Denote $C = [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$. Then $z'_i \in \text{co}\{d_i^1, d_i^2\}$, $i = 1, 2$, for every $z' \in U_X(x^1) \cap U_X(x^2) \cap C$.

We replace $B_X(x^1, r(x^1))$ with $V_X(x^1, r(x^1)) = B_X(x^1, r(x^1)) \setminus \text{cl}B_X(x^2, r(x^2))$ and keep $B_X(x^2, r(x^2))$ unchanged. Obviously, the open sets $V_X(x^1, r(x^1))$ and $B_X(x^2, r(x^2))$ do not intersect. However, in order to cover the compact set $A = \partial B_X(x^2, r(x^2)) \cap \text{cl}B_X(x^1, r(x^1))$, we have to add a finite number of new elements to the initial cover.

For every $x \in A$ with $x_2 \neq d_2^1$, we pick an open ball $B_X(x, r(x))$ such that $|x_2 - d_2^1| > 5r(x)$ and find a minimal $I(x) \subset I(x^1)$ (for $B_X(x, r(x))$) with $K(x) \subset K(x^1)$ such that for every $x' \in B_X(x, 3r(x))$, there exists $i \in I(x)$ with $u_i(d_i, x'_{-i}) > u_i(x')$.

For every $x \in A$ with $x_2 = d_2^1$, we pick $B_X(x, r(x))$ such that $|x'_1 - d_1^2| > 5r(x)$ and find a minimal $I(x) \subset I(x^2)$ with $K(x) \subset K(x^2)$ such that for every $x' \in B_X(x, 3r(x))$ there exists $i \in I(x)$ with $u_i(d_i, x'_{-i}) > u_i(x')$.

At first glance, it looks like not much has changed. However, it is not so. For example, if for some $x \in A \cap \text{int}C$, there is $z' \in B_X(x, r(x)) \cap B_X(x^2, r(x^2))$ such that $z'_i \in \text{co}\{d_i, d_i^2\}$, $i = 1, 2$, then, obviously, $I(x) = \{1, 2\}$. Moreover, as we have shown in the proof of Theorem 3, it is possible only if $d_i = x_i$ for some $i \in \{1, 2\}$ (Case 7). However, this is not the case by construction. Therefore, the minimal $I(x)$ is a one-element set.

Appendix B

Proof of Lemma 2

Fix $x \in X \setminus E_G$. Let $A(x)$ be the set of $\alpha \in \mathbb{R}^n$ such that $(x, \alpha) \in \text{clGr}G$. For each $i \in N$, define $\underline{u}_i : X \rightarrow \mathbb{R}$ by $\underline{u}_i(x_i, x_{-i}) = \liminf_{x'_{-i} \rightarrow x_{-i}} u_i(x_i, x'_{-i})$. By construction, \underline{u}_i is lower semicontinuous in x_{-i} . For each $i \in N$ and every $x_{-i} \in X_{-i}$, define $\delta_i : X_{-i} \rightarrow \mathbb{R}$ by $\delta_i(x_{-i}) = \sup_{y_i \in X_i} \underline{u}_i(y_i, x_{-i})$. It is clear that each δ_i , as the supremum of a collection of lower semicontinuous functions, is lower semicontinuous (see also Reny 1999, p. 1037).

Since G is better-reply secure, for each $\alpha = (\alpha_1, \dots, \alpha_N) \in A(x)$ there is $i(\alpha) \in N$ such that $\delta_{i(\alpha)}(x_{-i(\alpha)}) > \alpha_{i(\alpha)}$. Pick $\varepsilon_\alpha > 0$ and $r_\alpha > 0$ such that $\delta_{i(\alpha)}(x_{-i(\alpha)}) > \alpha'_{i(\alpha)} + \varepsilon_\alpha$ for all $\alpha' \in B_{\mathbb{R}^n}(\alpha, r_\alpha)$. We can say that player $i(\alpha)$ secures the neighborhood $B_{\mathbb{R}^n}(\alpha, r_\alpha)$ at x .

Since $A(x)$ is compact, the cover $\{B_{\mathbb{R}^n}(\alpha, r_\alpha) : \alpha \in A(x)\}$ contains a finite subcover $\{B_{\mathbb{R}^n}(\alpha^j, r_{\alpha^j}) : j = 1, \dots, k\}$. Let $\varepsilon(x) = \frac{1}{2} \min_{j \in \{1, \dots, k\}} \varepsilon_{\alpha^j}$. Denote by $J_i(x)$ the collection of all $j \in \{1, \dots, k\}$ such that player i secures $B_{\mathbb{R}^n}(\alpha^j, r_{\alpha^j})$ at x . Let $I(x) = \{i \in N : J_i(x) \neq \emptyset\}$ and $\bar{\alpha}_i = \max_{j \in J_i} (\alpha^j + r_{\alpha^j})$. Then, by the definition of the least upper bound, for each $i \in I(x)$ there exists $d_i \in X_i$ such that $\underline{u}_i(d_i, x_{-i}) > \bar{\alpha}_i + \varepsilon(x)$. From the lower semicontinuity of \underline{u}_i in x_{-i} , we deduce that $\underline{u}_i(d_i, x'_{-i}) > \bar{\alpha}_i + \varepsilon(x)$ for all x'_{-i} in some open neighborhood $U_{X_{-i}}(x_{-i})$ of x_{-i} .

We claim that there exists an open neighborhood $U_X(x)$ of x such that, for every $x' \in U_X(x)$, there is some $i \in I(x)$ with $u_i(d_i, z_{-i}) - \varepsilon(x) > u_i(x')$ for all

$z_{-i} \in U_{X_{-i}}(x_{-i})$. If it is not so, then one can construct a net $\{x^\beta\}$ converging to x such that, for each β and each $i \in I(x)$, $u_i(d_i, z_{-i}^\beta) - \varepsilon(x) \leq u_i(x^\beta)$ for some $z_{-i}^\beta \in U_{X_{-i}}(x_{-i})$. Since the payoff functions are bounded, there is no loss of generality in assuming that the net $\{u(x^\beta)\}$ converges to some $\alpha \in A(x)$. Then, for some $j \in \{1, \dots, k\}$, there exists $\hat{\beta}$ such that $u(x^\beta) \in B_{\mathbb{R}^n}(\alpha^j, r_{\alpha^j})$ for all $\beta \succeq \hat{\beta}$. Therefore, for some $i \in I(x)$, $u_i(d_i, x'_{-i}) \geq \underline{u}_i(d_i, x'_{-i}) > \bar{\alpha}_i + \varepsilon(x) > u_i(x^\beta) + \varepsilon(x)$ for all $x'_{-i} \in U_{X_{-i}}(x_{-i})$ and all $\beta \succeq \hat{\beta}$, a contradiction.

Proof of Lemma 5

The compactness of X implies that the open cover $\{U_x : x \in X\}$ of X contains a finite subcover $\{U_{x_j} : j \in J\}$, where J is a finite set. Let $\{G_{x_j} : j \in J\}$ be a closed refinement of $\{U_{x_j} : j \in J\}$ such that $G_{x_j} \subset U_{x_j}$ for each $j \in J$. For each $j \in J$, define a correspondence $F_j : X \rightrightarrows Y$ by

$$F_j(z) = \begin{cases} \cup_{\{s \in J : z \in U_{x_s}\}} F_{x_s}(z), & \text{if } z \in G_{x_j}, \\ Y, & \text{if } z \notin G_{x_j}. \end{cases}$$

Assumption (ii) implies that $\theta(z) \notin \text{co}F_j(z)$ for all $z \in G_{x_j}$. Each F_j has open lower sections in X since, for each $y \in Y$,

$$\begin{aligned} F_j^{-1}(y) &= \{z \in G_{x_j} : y \in (\cup_{\{s \in J : z \in U_{x_s}\}} F_{x_s}(z))\} \cup (X \setminus G_{x_j}) \\ &= (G_{x_j} \cap (\cup_{s \in J} (U_{x_s} \cap F_{x_s}^{-1}(y)))) \cup (X \setminus G_{x_j}) \\ &= \cup_{s \in J} (U_{x_s} \cap F_{x_s}^{-1}(y)) \cup (X \setminus G_{x_j}). \end{aligned}$$

Therefore $\bar{F} : X \rightrightarrows Y$ defined by $\bar{F}(z) = \cap_{j \in J} F_j(z)$ also has open lower sections. By construction, $\text{Dom} \bar{F} = X$ and \bar{F} is of class L_θ .

References

Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis, A Hitchhiker's Guide, 3rd ed. Berlin: Springer-Verlag (2006)

- Bagh, A.: Equilibrium in abstract economies without the lower semi-continuity of the constraint maps. *J Math Econ* **30**, 175-185 (1998)
- Barelli, P., Soza, I.: On the existence of Nash equilibria in discontinuous and qualitative games. University of Rochester, mimeo (2009)
- Baye, M.R., Tian G., Zhou, J.: Characterization of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs. *Rev Econ Stud* **60**, 935-948 (1993)
- Borglin, A., Keiding, H.: Existence of equilibrium actions and of equilibrium: a note on the 'new' existence theorems. *J Math Econ* **3**, 313-316 (1976)
- Browder, F.: The fixed point theory of multi-valued mappings in topological vector spaces. *Math Ann* **177**, 283-301 (1968)
- Carmona, G.: Understanding some recent existence results for discontinuous games. *Econ. Theory*, forthcoming (2011)
- de Castro, L.I.: Equilibria existence in regular discontinuous games. *Econ. Theory*, forthcoming (2011)
- Ding, X.P., Kim, W.K., Tan, K.K.: Equilibria of generalized games with L -majorized correspondences. *Internat J Math & Math Sci* **17**, 783-790 (1994)
- Ding, X.P., Tan, K.K.: On equilibria of non-compact generalized games. *J Math Anal Appl* **177**, 226-238 (1993)
- Ding, X.P., Xia, F.Q.: Equilibria of nonparacompact generalized games with L_{FC} -majorized correspondences in G -convex spaces. *Nonlinear Analysis* **56**, 831-849 (2004)
- Lassonde, M., Schenkel, C.: KKM principle, fixed points, and Nash equilibria. *J Math Anal Appl* **164**, 542-548 (1992)
- McLennan, A., Monteiro, P.K., Tourky R.: Games with discontinuous payoffs: a strengthening of Reny's existence theorem. *Mimeo* (2009)
- Nessah, R., Tian, G.: Existence of equilibrium in discontinuous games. *Mimeo* (2010)
- Prokopovych, P.: On equilibrium existence in payoff secure games. *Econ. Theory*, forthcoming (2011)
- Reny, P.J.: On the existence of pure and mixed strategy Nash equilibria in dis-

continuous games. *Econometrica* **67**, 1029-1056 (1999)

Reny, P.J.: Further results on the existence of Nash equilibria in discontinuous games. University of Chicago, mimeo (2009)

Yannelis, N.C., Prabhakar, N.D.: Existence of maximal elements and equilibria in linear topological spaces. *J Math Econ* **12**, 233-245 (1983)

Yannelis, N.C. : The core of an economy without ordered preferences. In: Khan, M.A., Yannelis, N.C. (eds.) *Equilibrium Theory in Infinite Dimensional Spaces*, Berlin: Springer (1991)

Yuan, X.Z.: The study of equilibria for abstract economies in topological vector spaces – a unified approach. *Nonlinear Analysis* **37**, 409-430 (1999)